

THE MIXED HODGE STRUCTURE ON THE FUNDAMENTAL GROUP OF A PUNCTURED RIEMANN SURFACE

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ABSTRACT. Given a compact Riemann surface \bar{X} of genus g and a point q on \bar{X} , we consider $X := \bar{X} \setminus \{q\}$ with a basepoint $p \in X$. The extension of mixed Hodge structures, given by the weights -1 and -2, of the mixed Hodge structure on the fundamental group (in the sense of Hain) is studied. We show that it naturally corresponds on the one hand to the element $(2gq - 2p - K)$ in $\text{Pic}^0(\bar{X})$, where K represents the canonical divisor, and on the other hand to the respective extension of \bar{X} . Finally, we prove a pointed Torelli theorem for punctured Riemann surfaces.

INTRODUCTION

Let q be a point in a compact Riemann surface \bar{X} of genus g . In this paper we want to study the complement of q in \bar{X} , i. e. $X := \bar{X} \setminus \{q\}$ with a basepoint p . We refer to this situation as to a *punctured Riemann surface X with puncture $q \in \bar{X}$ and basepoint p* .

For compact Riemann surfaces, Hain and Pulte ([Hai87b], [Pul88]) proved that the extension of mixed Hodge structure, which is given by the weights -1 and -2 of the mixed Hodge structure (MHS) on the fundamental group, determines the base point (see Theorem 2.1). From this result and from the classical Torelli theorem they derived a *pointed Torelli theorem* (see Theorem 2.3) as a corollary.

For a punctured Riemann surface $(\bar{X} \setminus \{q\}, p)$, the extension w_{pq} given by the weights -1 and -2 of the MHS on the fundamental group is one dimension bigger than in the compact case. We show that it naturally corresponds on the one hand to the element

$$(2gq - 2p - K) \text{ in } \text{Pic}^0(\bar{X}),$$

where K represents the canonical divisor, and on the other hand to the respective extension of \bar{X} (see Theorem 1.2). This may have implications on possible normal functions on the moduli space of complex projective curves (cf. [HL97], 7.4).

Finally we prove that this extension w_{pq} determines both, the basepoint p and the ‘removed point’ q . This, together with the pointed Torelli theorem of Hain and Pulte yields a *punctured pointed Torelli theorem* (see Theorem 2.8).

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1. EXTENSIONS AND THE THETA DIVISOR

For the definition of iterated integrals and of the MHS on the fundamental group we refer to the introductory article [Hai87b].

Let \bar{X} be a compact Riemann surface of genus g and let q be a point on \bar{X} . We consider the pointed space (X, p) , where $X := \bar{X} \setminus \{q\}$ and p is a basepoint on X . Denote by $J \subset \mathbb{Z}\pi_1(X, p)$ and $\bar{J} \subset \mathbb{Z}\pi_1(\bar{X}, p)$ the augmentation ideals in the group rings of the respective fundamental groups. Note that $J/J^2 = H_1(X)$ resp. $\bar{J}/\bar{J}^2 = H_1(\bar{X})$, and since we remove only a single point from \bar{X} , we have that $X \hookrightarrow \bar{X}$ induces an isomorphism of pure Hodge structures between $H_1(X)$ and $H_1(\bar{X})$, both of weight -1. This allows us to identify these two Hodge structures and their duals. We will write just H_1 and H^1 . The *MHS on the fundamental group* $\pi_1(X, p)$ resp. $\pi_1(\bar{X}, p)$ consists by definition of MHSs on the integral lattices J/J^{s+1} resp. \bar{J}/\bar{J}^{s+1} for $s \geq 2$.

This definition of the MHSs is possible because of Chen's π_1 -De Rham-Theorem, telling us that integration of iterated integrals yields isomorphisms

$$(1.1) \quad \begin{aligned} H^0 \bar{B}_s(E^\bullet(\bar{X} \log q), p) &\xrightarrow{\cong} \text{Hom}_{\mathbb{Z}}(J/J^{s+1}, \mathbb{C}) =: (J/J^{s+1})_{\mathbb{C}}^* \quad \text{resp.} \\ H^0 \bar{B}_s(E^\bullet(\bar{X}), p) &\xrightarrow{\cong} \text{Hom}_{\mathbb{Z}}(\bar{J}/\bar{J}^{s+1}, \mathbb{C}) =: (\bar{J}/\bar{J}^{s+1})_{\mathbb{C}}^*. \end{aligned}$$

Here $E^\bullet(\bar{X} \log q)$ denotes the differential graded algebra (dga) of C^∞ -forms on $X = \bar{X} \setminus \{q\}$ with logarithmic singularities at q and $E^\bullet(\bar{X})$ denotes the dga of smooth complex valued forms on \bar{X} . The objects on the left of (1.1) are the complex vector spaces of iterated integrals of length $\leq s$, which are homotopy functionals – considered as functions on the fundamental group. These vector spaces can be described purely algebraically in terms of the augmented dga's $E^\bullet(\bar{X} \log q)$ and $E^\bullet(\bar{X})$. This is part of a general construction, *the reduced bar construction*, whence the elaborate notation (cf. [Che76] or [Hai87a]). Here we identify these different descriptions deliberately.

In the two cases under consideration, the weight filtration W_\bullet is already given on the lattices J/J^{s+1} resp. \bar{J}/\bar{J}^{s+1} by the J -adic filtration, i. e.

$$W_{-l} J/J^{s+1} = J^l/J^{s+1} \text{ resp. } W_{-l} \bar{J}/\bar{J}^{s+1} = \bar{J}^l/\bar{J}^{s+1} \text{ for } l > 0.$$

For $l = 1$ and $s = 1$ we recover the pure Hodge structure on homology, i. e. $W_{-1} J/J^2 = J/J^2 = H_1 = \bar{J}/\bar{J}^2 = W_{-1} \bar{J}/\bar{J}^2$. The weights -1 and -2 give rise to two extensions of MHSs w_{pq} and w_p , related by the following commutative diagram

$$(1.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & J^2/J^3 & \longrightarrow & J/J^3 & \longrightarrow & J/J^2 \longrightarrow 0 & ; w_{pq} \\ & & \downarrow & & \downarrow & & \downarrow = & \\ 0 & \longrightarrow & \bar{J}^2/\bar{J}^3 & \longrightarrow & \bar{J}/\bar{J}^3 & \longrightarrow & \bar{J}/\bar{J}^2 \longrightarrow 0 & ; w_p. \end{array}$$

The multiplication in the group rings defines surjective maps $J/J^2 \otimes J/J^2 \rightarrow J^2/J^3$ and $\bar{J}/\bar{J}^2 \otimes \bar{J}/\bar{J}^2 \rightarrow \bar{J}^2/\bar{J}^3$ whose dual morphisms are inclusions $(J^2/J^3)^* \hookrightarrow H^1 \otimes H^1$ and $(\bar{J}^2/\bar{J}^3)^* \hookrightarrow H^1 \otimes H^1$. It is well-known (cf. [Hai87b]) that in both cases, the image of the above inclusions coincides with the kernel of the cup-product. Hence the inclusions give isomorphisms $(J^2/J^3)^* \cong H^1 \otimes H^1$ and $(\bar{J}^2/\bar{J}^3)^* \cong K$, where $K := \ker\{\cup : H^1(\bar{X}) \otimes H^1(\bar{X}) \rightarrow H^2(\bar{X})\}$. As \cup is a morphism of Hodge structures, K inherits a pure Hodge structure of weight 2 from $H^1 \otimes H^1$. Dualizing

the diagram (1.2) yields extensions of MHSs m_{pq} and m_p – dual to w_{pq} and w_p

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & (J/J^3)^* & \longrightarrow & H^1 \otimes H^1 \longrightarrow 0 & ; m_{pq} \\ & & \uparrow & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & H^1 & \longrightarrow & (\bar{J}/\bar{J}^3)^* & \longrightarrow & K \longrightarrow 0 & ; m_p. \end{array}$$

Since K is not a direct summand of $H^1 \otimes H^1$ over \mathbb{Z} , let us first clarify the nature of the embedding $K \hookrightarrow H^1 \otimes H^1$. Identify $H^2(\bar{X}, \mathbb{Z})$ with \mathbb{Z} . There is a bilinear form

$$b : (H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1) \times (H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1) \longrightarrow \mathbb{Z},$$

given by $b((x_1 \otimes x_2), (y_1 \otimes y_2)) := (x_1 \cup y_2) \cdot (y_1 \cup x_2)$, which has mixed signature and is non degenerate. Consider the rank 1 sublattice of $H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1$ orthogonal to the kernel of the cup-product $K_{\mathbb{Z}} \subset H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1$ with respect to b . The following commutative diagram describes how $Q_{\mathbb{Z}}$ is related to $K_{\mathbb{Z}} \subset H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1$ and $H^2(\bar{X}, \mathbb{Z})$.

$$\begin{array}{ccc} Q_{\mathbb{Z}} \oplus K_{\mathbb{Z}} & \hookrightarrow & H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1 \\ \frac{1}{2g} \cup \downarrow & & \downarrow \cup \\ H^2(\bar{X}, \mathbb{Z}) & \xrightarrow{\cdot 2g} & H^2(\bar{X}, \mathbb{Z}). \end{array}$$

$Q_{\mathbb{Z}}$ is generated by one element \mathfrak{X} in $H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1$, which is invariant under complex conjugation. Hence $H_{\mathbb{Z}}^1 \otimes H_{\mathbb{Z}}^1$ induces on $Q_{\mathbb{Z}}$ a \mathbb{Z} -HS, isomorphic to $H^2(\bar{X}, \mathbb{Z})$ or $\mathbb{Z}(-1)$. Note that over the rationals holds: $K_{\mathbb{Q}} \oplus Q_{\mathbb{Q}} = H_{\mathbb{Q}}^1 \otimes H_{\mathbb{Q}}^1$.

Definition 1.1. Define $k_{pq} = [0 \rightarrow H^1 \rightarrow E_{pq} \rightarrow Q \rightarrow 0] \in \text{Ext}_{\text{MHS}}(Q; H^1)$ to be the restriction of the extension m_{pq} to an extension of Q by H^1 .

Let $\Psi : \text{Ext}_{\text{MHS}}(Q; H^1) \xrightarrow{\cong} \text{Pic}^0(\bar{X})$ be the natural isomorphism (see [Car80]), then the main theorem of this paper is

Theorem 1.2. $\Psi(k_{pq}) = (2gq - 2p - K) \in \text{Pic}^0(\bar{X})$.

1.1. Riemann's Constant. Let $u : \text{Pic}^0(\bar{X}) \rightarrow \text{Jac}(\bar{X})$ be the Abel-Jacobi map and define the divisor

$$W_{p,g-1} := \left\{ \sum_{j=1}^{g-1} u(q_j - p) \mid \sum_{j=1}^{g-1} q_j \in \bar{X}^{(g-1)} \right\}.$$

Denote the *theta divisor* on $\text{Jac}(\bar{X})$ by Θ and *Riemann's constant* by $\kappa_p \in \text{Jac}(\bar{X})$, such that Riemann's classical theorem¹ reads: $\Theta = W_{p,g-1} + \kappa_p$.

Using the Riemann-Roch theorem one can prove that Riemann's constant κ_p is related to the canonical divisor by the fact that for any divisor K of a holomorphic 1-form holds $u((2g-2)p - K) = 2\kappa_p$ and that the canonical divisor is characterized by this equation (for a proof we refer to [GH78], p. 340). Theorem 1.2 is then a consequence of the following theorem, whose proof will be given in the sequel.

Theorem 1.3. $u(\Psi(k_{pq}) + 2g(p - q)) = 2\kappa_p$.

¹Proofs of this theorem can be found in [Rie92] (VI, 22., pp. 132-136; XI, pp. 213-224) or [Lan02]. For proofs in modern language we refer to [Lew64], [Mum83] (Theorem 3.1, pp. 149-151) or to [GH78]. In the theory of θ -functions it is more convenient to define \varkappa_p in \mathbb{C}^g like [Rie92], [Lan02], [Lew64], [Fay73] (here Riemann's constant is defined as $-\varkappa_p$) and [Mum83] do.

The rest of this section is devoted to the proof of Theorem 1.3. First we interpret the right hand side of the equation by means of an expression for κ_p in terms of iterated integrals, as it was already known to Riemann.

To present this formula we need some more notation. Denote by $\gamma_1, \dots, \gamma_{2g}$ and δ a system of elements in $\pi_1(X, p)$ having the property, that the fundamental group $\pi_1(X, p)$ is the quotient of the free group $F\langle \gamma_1, \dots, \gamma_{2g}, \delta \rangle$ generated by the γ_i and δ subject to the commutator relation

$$(1.3) \quad [\gamma_1, \gamma_{g+1}] \cdots [\gamma_g, \gamma_{2g}] = \delta.$$

Let dz_1, \dots, dz_g be a basis of holomorphic 1-forms on \bar{X} , such that $\int_{\gamma_\nu} dz_i = \delta_{i\nu}$, i. e. the period matrix can be written $\Omega = (\omega_{i\mu})_{\substack{i=1, \dots, g \\ \mu=1, \dots, 2g}} = (\Omega_1, \Omega_2) = (I, Z)$. By Riemann's bilinear relations, Z is a symmetric $g \times g$ -matrix with positive definite imaginary part. Having made these choices we may represent the Jacobian of \bar{X} as $\text{Jac}(\bar{X}) := \mathbb{C}^g / \Omega \mathbb{Z}^{2g}$. The following expression of κ_p in terms of iterated integrals

$$(1.4) \quad \kappa_p = \left[- \sum_{\nu=1}^g \int_{\gamma_\nu} dz_i dz_\nu + \frac{1}{2} \int_{\gamma_{g+i}} dz_i \right]_{i=1, \dots, g} \in \text{Jac}(\bar{X})$$

was known to Bernhard Riemann in 1865 (see [Rie92], p. 213 or [Fay73]).

1.2. Extension Data. According to [Car80] we need two things for the computation of the extension data $k_{pq} \in \text{Ext}_{\text{MHS}}(Q, H^1)$: a Hodge filtration preserving section $s_F : (Q, F^\bullet) \rightarrow (E_{pq}, F^\bullet)$ and an integral retraction $r_{\mathbb{Z}} : (E_{pq})_{\mathbb{Z}} \rightarrow H_{\mathbb{Z}}^1$.

Let dx_1, \dots, dx_{2g} be the real harmonic 1-forms such that $\int_{\gamma_j} dx_i = \delta_{ij}$. Then a generator \mathfrak{X} of $Q_{\mathbb{Z}}$ is given by $\mathfrak{X} := \sum_{\nu=1}^g ([dx_\nu] \otimes [dx_{g+\nu}] - [dx_{g+\nu}] \otimes [dx_\nu])$. Riemann's first bilinear relation tells us that $\mathfrak{X} \in F^1(H_{\mathbb{C}}^1 \otimes H_{\mathbb{C}}^1)$. To be more precise, we can write

$$\sum_{\nu=1}^g (dx_\nu \otimes dx_{g+\nu} - dx_{g+\nu} \otimes dx_\nu) = \sum_{j,k=1}^g (a_{jk} dz_j \otimes d\bar{z}_k + \bar{a}_{jk} d\bar{z}_j \otimes dz_k)$$

with $A = (a_{jk})_{jk} = (\bar{\Omega}_2 \Omega_1^t - \bar{\Omega}_1 \Omega_2^t)^{-1} = (\bar{Z} - Z)^{-1}$. Observe: $A^t = -\bar{A}$.

Define $\wedge \mathfrak{X} := \sum_{\nu=1}^g (dx_\nu \wedge dx_{g+\nu} - dx_{g+\nu} \wedge dx_\nu) \in F^1 E^2(\bar{X})$. The strictness of the differential with respect to the Hodge filtration on $E^\bullet(\bar{X} \log q)$ implies that there is a $\mu_q \in F^1 E^1(\bar{X} \log q)$ such that $\wedge \mathfrak{X} + d\mu_q = 0$. This condition implies that the iterated integral $\int \mathfrak{X} + \mu_q := \int \sum_{\nu=1}^g (dx_\nu dx_{g+\nu} - dx_{g+\nu} dx_\nu) + \mu_q$ is a homotopy functional.

A Hodge filtration preserving section s_F is then defined by $s_F(\mathfrak{X}) = \int \mathfrak{X} + \mu_q$ and an integral retraction $r_{\mathbb{Z}}$ is given by the map, which sends an iterated integral $\int I$ of length ≤ 2 with values in \mathbb{Z} to $r_{\mathbb{Z}}(\int I) := \sum_{j=1}^{2g} (\int_{\gamma_j} I) [dx_j]$. Again a standard computation shows

$$u \circ \Psi(k_{pq}) = \left(\sum_{\nu=1}^g \left(\int_{\gamma_\nu} dz_i \int_{\gamma_{g+\nu}} \mathfrak{X} + \mu_q - \int_{\gamma_{g+\nu}} dz_i \int_{\gamma_\nu} \mathfrak{X} + \mu_q \right) \right)_{i=1, \dots, g} \in \text{Jac } \bar{X}.$$

1.3. A Higher Reciprocity Law. Generally for functions $F, G : \pi_1(X, p) \rightarrow \mathbb{C}$ we introduce: $\Pi(F; G) := \sum_{\nu=1}^g (F(\gamma_\nu) G(\gamma_{g+\nu}) - F(\gamma_{g+\nu}) G(\gamma_\nu))$. For instance, Riemann's classical period relation reads: $\Pi(\int dz_i; \int dz_j) = 0$. With this notation we can state a higher reciprocity law.

Theorem 1.4. *For any holomorphic 1-form ω on \bar{X} holds:*

$$\sum_{\nu=1}^g \left\{ \int_{\gamma_\nu} \omega \int_{\gamma_{g+\nu}} \mathfrak{X} + \mu_q - \int_{\gamma_{g+\nu}} \omega \int_{\gamma_\nu} \mathfrak{X} + \mu_q \right\} = 2g \int_p^q \omega$$

$$+ \sum_{j,k=1}^g a_{jk} \left\{ \Pi \left(\int \omega; \int dz_j \int d\bar{z}_k \right) + 2 \Pi \left(\int \omega \int d\bar{z}_k; \int dz_j \right) - 2 \Pi \left(\int \omega dz_j; \int d\bar{z}_k \right) \right\}.$$

1.3.1. *Observation.* The proof of the *higher reciprocity law* in Theorem 1.4 and also later the proof of the *higher period relation* of Theorem 1.5 are direct generalizations of Riemann's bilinear relations as they are proved in [Che77] or [Gun69]. Similar considerations like the following can also be found in [PY96].

Let $c_i := (\gamma_i - 1)$ and $d := (\delta - 1)$ denote the elements in J corresponding to γ_i and δ in $\pi_1(X, p)$. If we interpret relation (1.3) in $\mathbb{Z}\pi_1(X, p)$ modulo J^4 , we obtain

$$(1.5) \quad \sum_{\nu=1}^g \{ c_\nu c_{g+\nu} - c_{g+\nu} c_\nu$$

$$+ (c_{g+\nu} c_\nu c_{g+\nu} - c_\nu c_{g+\nu} c_\nu) - (c_\nu c_{g+\nu} c_{g+\nu} - c_{g+\nu} c_\nu c_\nu) \} \equiv d \pmod{J^4}.$$

When the linear extension of a homotopy functional $F : \pi_1(X, p) \rightarrow \mathbb{C}$ to $\mathbb{Z}\pi_1(X, p)$ satisfies $F(J^4) = 0$, then it has to respect this relation. For instance iterated integrals of length ≤ 3 , which are homotopy functionals, are examples for such F .

Proof of Theorem 1.4: Use that for a closed path α holds: $\int_\alpha d\bar{z}_j dz_k + \int_\alpha dz_k d\bar{z}_j = \int_\alpha d\bar{z}_j \int_\alpha dz_k$ to prove that the left hand side of the equation in Theorem 1.4 equals

$$\Pi \left(\int \omega; \int \sum_{j,k=1}^g 2a_{jk} dz_j d\bar{z}_k + \mu_q \right) - \Pi \left(\int \omega; \sum_{j,k=1}^g a_{jk} \int dz_j \int d\bar{z}_k \right).$$

Note that $\int I := \int \sum_{j,k=1}^g 2a_{jk} \omega dz_j d\bar{z}_k + \omega \mu_q$ is a homotopy functional, so its values on both sides of (1.5) coincide. Recall that for 1-forms φ, ψ, χ and closed paths α, β with $a = (\alpha - 1)$, $b = (\beta - 1)$ and $ab = (\alpha\beta - \alpha - \beta + 1)$ holds: $\int_{ab} \varphi \psi \chi = \int_a \varphi \int_b \psi \chi + \int_a \varphi \psi \int_b \chi$. Using this rule, a direct computation shows that the value of $\int I$ on the left hand side of relation (1.5) takes the value:

$$\Pi \left(\int \omega; \int I \right)$$

$$+ \sum_{j,k=1}^g 2a_{jk} \left\{ \Pi \left(\int \omega dz_j; \int d\bar{z}_k \right) - \Pi \left(\int \omega \int d\bar{z}_k; \int dz_j \right) - \Pi \left(\int \omega; \int dz_j \int d\bar{z}_k \right) \right\}.$$

According to our observation 1.3.1 this has to be equal to the value of the homotopy functional $\int I$ applied to the right hand side of (1.5). We compute this value as follows. From $\wedge \mathfrak{X} + d\mu_q = 0$ we can determine the shape of μ_q . Using Stokes' theorem, a standard argument shows that there is a simply connected holomorphic coordinate plot (U, z) on \bar{X} containing q and all of a representing path for $\delta \in \pi_1(X, p)$ such that on U we may write $\mu_q = \frac{2g}{2\pi i} \frac{dz}{z} + \varphi$, where φ is a smooth (non-closed) 1-form in $E^1(U)$. Since this representative of δ is 0-homotopic in U , the homotopy functional $\sum_{j,k=1}^g 2a_{jk} \int \omega dz_j d\bar{z}_k + \omega \varphi$ vanishes on it. It remains: $\int_\delta \omega \left(\frac{2g}{2\pi i} \frac{dz}{z} \right) = 2g \int_p^q \omega$. Putting all ingredients together provides the proof. \square

1.4. A Higher Period Relation. Recall that we chose $\Omega = (I, Z)$ with symmetric Z having positive imaginary part and with $A = (\bar{Z} - Z)^{-1}$. Define for $i = 1, \dots, g$ the $g \times g$ -matrices

$$I_1^i := \left(\int_{c_\nu} dz_i dz_j \right)_{\nu,j} \quad \text{and} \quad I_2^i := \left(\int_{c_{g+\nu}} dz_i dz_j \right)_{\nu,j} \in \text{Mat}(g \times g; \mathbb{C}).$$

Then we define the following two vectors with entries in $\text{Mat}(g \times g; \mathbb{C})$

$$I_1 = \begin{pmatrix} I_1^1 \\ \vdots \\ I_1^g \end{pmatrix}, \quad I_2 = \begin{pmatrix} I_2^1 \\ \vdots \\ I_2^g \end{pmatrix} \in \text{Mat}(g \times 1; \text{Mat}(g \times g)).$$

For some matrix M , denote by $\text{tr } M$ the *trace* of M and by $\text{diag } M$ its *diagonal*. For a vector of matrices let *the trace* of this vector be the vector of the traces of the matrices. The following theorem is the announced *higher period relation*.

Theorem 1.5.

$$\begin{aligned} & (2 \text{tr}(I_2 A) - 2 \text{tr}(I_1 A Z)) + (\text{diag}(Z A Z) - Z \text{diag}(A Z)) \\ & + (\text{diag}(Z A) - Z \text{diag}(A)) + (\text{diag}(A Z) - Z \text{diag}(Z A)) \equiv 0 \pmod{(I, Z) \mathbb{Z}^{2g}}. \end{aligned}$$

Proof: Apply the homotopy functional $\sum_{j,k=1}^g a_{jk} \int dz_j dz_i dz_k$ to (1.5). \square

With the above notation, we use this higher period relation to continue our computation of the extension k_{pq} . After Theorem 1.4 it makes sense to speak of k_{pq} ; we have $\Psi(k_{pq}) = 2g(q-p) + \Psi(k_{pp})$. For $u \circ \Psi(k_{pp}) \in \mathbb{C}^g / \Omega \mathbb{Z}^{2g}$ we had the following expression:

$$\text{diag}(Z A \bar{Z}) - Z \text{diag}(A) + 2 \text{diag}(Z A) - 2 Z \text{diag}(A \bar{Z}) - 2 \text{tr}(I_1 A \bar{Z}) + 2 \text{tr}(I_2 A)$$

Transform this expression such that it only contains (iterated) integrals over *holomorphic* forms. Observe $\text{diag}(Z A \bar{Z}) = \text{diag}(Z(\bar{Z} - Z)^{-1}(\bar{Z} - Z)) + \text{diag}(Z A Z) = \text{diag}(Z) + \text{diag}(Z A Z)$ and similarly $2 Z \text{diag}(A \bar{Z}) \equiv 2 Z \text{diag}(A Z) \pmod{(I, Z) \mathbb{Z}^{2g}}$ and $2 \text{tr}(I_1 A \bar{Z}) = 2 \text{tr}(I_1) + 2 \text{tr}(I_1 A Z)$. Using these identities we continue

$$\begin{aligned} u \circ \Psi(k_{pp}) & \equiv \text{diag}(Z) + \text{diag}(Z A Z) - Z \text{diag}(A) \\ & + 2 \text{diag}(Z A) - 2 Z \text{diag}(A Z) \\ & - 2 \text{tr}(I_1) - 2 \text{tr}(I_1 A Z) + 2 \text{tr}(I_2 A) \pmod{(I, Z) \mathbb{Z}^{2g}}. \end{aligned}$$

Notice: $\text{diag}(Z A) - \text{diag}(A Z) = 0$. When we apply Theorem 1.5, we finally get: $u \circ \Psi(k_{pp}) \equiv \text{diag } Z - 2 \text{tr}(I_1) \pmod{(I, Z) \mathbb{Z}^{2g}}$. Writing this out, we find $u \circ \Psi(k_{pp}) \equiv 2\kappa_p \pmod{(I, Z) \mathbb{Z}^{2g}}$ by virtue of formula (1.4). This is the proof of Theorem 1.3.

2. A POINTED TORELLI THEOREM FOR PUNCTURED RIEMANN SURFACES

Here we want to show that the extension w_{pq} or respectively m_{pq} determines p and q . For this we do not need Section 1. Finally we will combine this with results of Hain and Pulte [Hai87b], [Pul88], which we briefly sketch first.

2.1. The Pointed Torelli Theorem. The pointed Torelli Theorem of Hain and Pulte is based on the following.

Theorem 2.1 (Hain, Pulte). *The map from $\text{Pic}^0 \bar{X}$ to $\text{Ext}_{\text{MHS}}(K; H^1)$ which maps $(p - p')$ to $m_p - m_{p'}$ is well-defined and injective.*

We write $(\bar{X}, p) \cong (\bar{X}, p')$ if there is an automorphism $\phi : \bar{X} \rightarrow \bar{X}$ that maps p to p' . For a point p on \bar{X} we define the set of alternatives for p as

$$a_{\bar{X}}(p) := \{p\} \cup \{p' \in \bar{X} \mid m_{p'} = -m_p \text{ and } (\bar{X}, p) \not\cong (\bar{X}, p')\}$$

The following is a consequence of Theorem 2.1. Let us give a short proof of it.

Corollary 2.2. $a_{\bar{X}}(p)$ consists of at most two points. Up to automorphism of \bar{X} , there cannot be more than one pair of different points $\{p, p'\}$ on \bar{X} such that $a_{\bar{X}}(p) = \{p, p'\} = a_{\bar{X}}(p')$.

Proof: The first assertion is an obvious consequence of 2.1. To prove the second assertion, assume that \tilde{p} and \tilde{p}' is another such pair with $a_{\bar{X}}(\tilde{p}) = \{\tilde{p}, \tilde{p}'\} = a_{\bar{X}}(\tilde{p}')$. Then by 2.1, the divisors $p + p' = \tilde{p} + \tilde{p}'$ are linearly equivalent. It follows that either $\{p, p'\} = \{\tilde{p}, \tilde{p}'\}$ or \bar{X} is hyperelliptic and the hyperelliptic involution maps p to p' and q to q' , which contradicts the assumptions on p, p' and q, q' . \square

Together with the classical Torelli theorem, Hain and Pulte used Theorem 2.1 to prove the following *pointed Torelli theorem*. For a pointed compact Riemann surface (\bar{Z}, z_0) denote by $J_{z_0}(\bar{Z})$ the augmentation ideal in $\mathbb{Z}\pi_1(\bar{Z}, z_0)$.

Theorem 2.3 (Hain, Pulte). *Suppose that (\bar{X}, p) and (\bar{Y}, r) are two pointed compact Riemann surfaces. If there is a ring homomorphism*

$$\mathbb{Z}\pi_1(\bar{X}, p) / J_p(\bar{X})^3 \xrightarrow{\cong} \mathbb{Z}\pi_1(\bar{Y}, r) / J_r(\bar{Y})^3$$

which induces an isomorphism of MHSs, then there is an isomorphism $f : \bar{X} \rightarrow \bar{Y}$ with $f(p) \in a_{\bar{Y}}(r)$.

Remark 2.4. As far as the author knows, still no example is known of a pointed compact Riemann surface (\bar{X}, p) with $|a_{\bar{X}}(p)| = 2$. M. Pulte [Pul88] has shown that such an (\bar{X}, p) with $|a_{\bar{X}}(p)| = 2$ must have zero harmonic volume. B. Harris [Har83] proved that a generic smooth projective complex curve has non zero harmonic volume. Moreover, Pulte showed (loc. cit.) that, if there are two points p, p' with $a_{\bar{X}}(p) = \{p, p'\} = a_{\bar{X}}(p')$, then $(g - 1)(p + p') - K = 0 \in \text{Pic}^0 \bar{X}$, where K is the canonical divisor. For pointed hyperelliptic curves (\bar{X}, p) always holds: $a_{\bar{X}}(p) = \{p\}$, since here $m_p = -m_{p'}$ implies $(\bar{X}, p) \cong (\bar{X}, p')$ by the hyperelliptic involution.

2.2. A Punctured Pointed Torelli Theorem. The following Theorem will follow directly from Lemma 2.9, which we prove at the end of this section.

Theorem 2.5. *For all $p \in \bar{X}$, the map $\text{Pic}^0 \bar{X} \rightarrow \text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1)$ which maps $(q - q')$ to $m_{pq} - m_{pq'}$ is well-defined and injective.*

Combining Theorem 2.5 with the results of Hain and Pulte we find.

Proposition 2.6. *The map from $(\bar{X} \times \bar{X}) \setminus \Delta$ to $\text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1)$ given by $(p, q) \mapsto m_{pq}$ is well-defined, extends to the diagonal Δ and is injective.*

Proof of 2.6: Note that the map of complex tori $\text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1) \rightarrow \text{Ext}_{\text{MHS}}(K \oplus Q; H^1)$ is a covering map, since $\text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{Z}} \hookrightarrow \text{Hom}(K \oplus Q; H^1)_{\mathbb{Z}}$. Moreover, we have the commutative diagram

$$\begin{array}{ccc} (X \times X) \setminus \Delta & \xrightarrow{\tilde{\varphi}} & \text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1) \\ \downarrow & & \downarrow \text{covering map} \\ X \times X & \xrightarrow{\varphi} & \text{Ext}_{\text{MHS}}(K; H^1) \oplus \text{Pic}^0 \bar{X} \\ (p, q) & \mapsto & (m_p, (2gq - 2p - K)). \end{array}$$

The map φ is continuous (m_p is – in a coordinate system – an expression of iterated integrals over paths with basepoint p). As the map $\tilde{\varphi}(p, q) = m_{pq}$ is a lifting of φ , we see that $\tilde{\varphi}$ is continuous too. The fact that the map $m_{pq} \mapsto (m_p, k_{pq})$ is a covering map tells us moreover that we may extend $\tilde{\varphi}$ to the diagonal Δ . Now by 2.1, the result of Hain and Pulte, the extension m_{pq} determines p . By Theorem 2.5 it determines also q . \square

Pulling back the intersection form $H_1(\bar{X}, \mathbb{Z}) \otimes H_1(\bar{X}, \mathbb{Z}) \rightarrow \mathbb{Z}$ along the natural isomorphism $J/J^2 \xrightarrow{\cong} \bar{J}/\bar{J}^2$ induces a polarization on $\text{Gr}_{-1}^W(J/J^3) = H^1(X)$. We can also put a polarization on $\text{Gr}_{-2}^W(J/J^3) = J^2/J^3 \cong J/J^2 \otimes J/J^2 = \text{Gr}_{-1}^W(J/J^3) \otimes \text{Gr}_{-1}^W(J/J^3)$, by taking the tensor product of the polarized Hodge structure $H^1(X)$ in the category of polarized Hodge structures. In that sense, J/J^3 becomes a *graded polarized MHS*, i. e. each Gr_l^W is a polarized Hodge structure.

For points p and q on \bar{X} we define

$$A_{\bar{X}}(p, q) := \{(p, q)\} \cup \left\{ (p', q') \in \bar{X} \times \bar{X} \mid \begin{array}{l} m_{p'q'} = -m_{pq} \text{ and} \\ (\bar{X} \setminus \{q\}, p) \not\cong (\bar{X} \setminus \{q'\}, p') \end{array} \right\}.$$

The following is then a consequence of Proposition 2.6.

Corollary 2.7. $A_{\bar{X}}(p, q)$ consists of at most two elements. \square

Our results lead to the following *punctured pointed Torelli theorem*.

Theorem 2.8. *Suppose that $(\bar{X} \setminus \{q\}, p)$ and $(\bar{Y} \setminus \{s\}, r)$ are two punctured compact Riemann surfaces with basepoint. If there is a ring isomorphism*

$$\mathbb{Z}\pi_1(\bar{X} \setminus \{q\}, p) / J_p(\bar{X} \setminus \{q\})^3 \xrightarrow{\cong} \mathbb{Z}\pi_1(\bar{Y} \setminus \{s\}, r) / J_r(\bar{Y} \setminus \{s\})^3,$$

which induces an isomorphism of graded polarized MHSs, then there is a biholomorphism $f : \bar{X} \rightarrow \bar{Y}$ with $(f(p), f(q)) \in A_{\bar{Y}}(r, s)$.

Proof of 2.8: The proof goes along the lines of the proof of the pointed Torelli theorem in [Pul88] and [Hai87b]. Let $J_{pq} = J_p(\bar{X} \setminus \{q\})$ and $J_{rs} = J_r(\bar{Y} \setminus \{s\})$. We have an isomorphism of MHSs, $\lambda : J_{pq}/J_{pq}^3 \xrightarrow{\cong} J_{rs}/J_{rs}^3$ and in particular, λ induces an isomorphism of polarized Hodge structures

$$\lambda^* : H^1(\bar{Y}) = W_1(J_{rs}/J_{rs}^3)^* \rightarrow W_1(J_{pq}/J_{pq}^3)^* = H^1(\bar{X}).$$

By the classical Torelli theorem (cf. for instance [Mar63]) we know that there is a biholomorphism $f : \bar{X} \rightarrow \bar{Y}$ such that $f^* : H^1(\bar{Y}) \rightarrow H^1(\bar{X})$ is $\pm \lambda^*$. Since λ respects the *ring* structure, the by λ induced map $(J_{rs}^2/J_{rs}^3)^* \rightarrow (J_{pq}^2/J_{pq}^3)^*$ is determined by $\lambda^* : H^1(\bar{Y}) \rightarrow H^1(\bar{X})$ and hence,

$$f^* : (J_{rs}^2/J_{rs}^3)^* = H^1(\bar{Y}) \otimes H^1(\bar{Y}) \rightarrow H^1(\bar{X}) \otimes H^1(\bar{X}) = (J_{pq}^2/J_{pq}^3)^*$$

is equal to $\lambda^* \otimes \lambda^*$. Without loss of generality, we may therefore assume that $(\bar{Y} \setminus \{s\}, r) = (\bar{X} \setminus \{q'\}, p')$ for two points p' and q' in \bar{X} and that the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & (J_{pq}/J_{pq}^3)^* & \longrightarrow & H^1 \otimes H^1 \longrightarrow 0 \\ & & \pm id \downarrow & & \downarrow \lambda^* & & \downarrow id \\ 0 & \longrightarrow & H^1 & \longrightarrow & (J_{p'q'}/J_{p'q'}^3)^* & \longrightarrow & H^1 \otimes H^1 \longrightarrow 0. \end{array}$$

It follows that $m_{pq} = \pm m_{p'q'}$. This means that there either is an automorphism $\phi : (\bar{X} \setminus \{q\}, p) \rightarrow (\bar{X} \setminus \{q'\}, p')$ or $A_{\bar{X}}(p, q) = \{(p, q); (p', q')\} = A_{\bar{X}}(p', q')$. In both cases, the identity map is the map with the desired properties. \square

2.3. A Technical Lemma. Theorem 2.5 is a consequence of the following

Lemma 2.9. *For any element $\sum_i (q_i - q'_i) \in \text{Pic}^0 \bar{X}$ holds:*

$$\sum_i (q_i - q'_i) = 0 \in \text{Pic}^0 \bar{X} \Leftrightarrow \sum_i (m_{pq_i} - m_{pq'_i}) = 0 \in \text{Ext}_{\text{MHS}}((H^1)^{\otimes 2}; H^1).$$

Proof: Consider the isomorphism (cf. [Car80]) from $\text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1)$ to

$$\text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}} / (F^0 \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}} + \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{Z}}),$$

where the image of an extension m_{pq} is $[\phi_{pq}]$ for a certain $\phi_{pq} \in \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}}$. On an element $[\varphi] \otimes [\psi] \in H^1 \otimes H^1$, the homomorphism ϕ_{pq} has the following property. There is a $\eta_q \in F^1 E^1(X \log q)$ such that $\varphi \wedge \psi + d\eta_q = 0$ and $\phi_{pq}([\varphi] \otimes [\psi]) = \sum_{j=1}^{2g} (\int_{\gamma_j} \varphi \psi + \mu_q) [dx_j]$. If $[\varphi] \otimes [\psi] \in K$ then η_q can be chosen in $F^1 E^1(X)$ and does not depend on q , which shows that $(\phi_{pq} - \phi_{pq'})$ is zero on K . Therefore it is determined by its value on one element of $(H^1 \otimes H^1) \setminus K$; for instance on $[dx_1] \otimes [dx_{g+1}]$.

Given a divisor $D = \sum_i (q_i - q'_i)$ define the homomorphism $\Phi_D := \sum_i (\phi_{pq_i} - \phi_{pq'_i}) : H^1 \otimes H^1 \rightarrow H^1$. We will derive a series of equivalences. First, we have:

$$\begin{aligned} \sum_i (m_{pq_i} - m_{pq'_i}) = 0 \in \text{Ext}_{\text{MHS}}(H^1 \otimes H^1; H^1) \\ \Leftrightarrow \Phi_D \in F^0 \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{C}} + \text{Hom}(H^1 \otimes H^1; H^1)_{\mathbb{Z}}. \end{aligned}$$

Now let $\mathbf{w} \in H^{0,1} \otimes H^{0,1}$ be such that $[dx_1] \otimes [dx_{g+1}] - \mathbf{w} \in F^1(H^1 \otimes H^1) = H^{1,0} \otimes H^1 + H^1 \otimes H^{1,0}$. Note that $H^{0,1} \otimes H^{0,1} \subset K$ and hence $\Phi_D(\mathbf{w}) = 0$. Moreover $H^{1,0} \otimes H^{1,0} \subset K$ and $\Phi_D(H^{1,0} \otimes H^{1,0}) = 0$. Therefore, we may continue the series of equivalences by:

$$\Leftrightarrow \Phi_D([dx_1] \otimes [dx_{g+1}] - \mathbf{w}) \in H^{1,0} + H_{\mathbb{Z}}^1 \Leftrightarrow \Phi_D([dx_1] \otimes [dx_{g+1}]) \in H^{1,0} + H_{\mathbb{Z}}^1.$$

Let $\eta_{q_i} \in F^1 E^1(X \log q_i)$ and $\eta_{q'_i} \in F^1 E^1(X \log q'_i)$ be such that $dx_1 \wedge dx_{g+1} + d\eta_{q_i} = 0$ and $dx_1 \wedge dx_{g+1} + d\eta_{q'_i} = 0$. Note that this implies $\text{Res}_{q_i} \eta_{q_i} = \frac{1}{2\pi i} = \text{Res}_{q'_i} \eta_{q'_i}$. Then a direct computation shows that we may go on:

$$\begin{aligned} \Leftrightarrow \sum_{j=1}^{2g} \sum_i \left(\int_{\gamma_j} \mu_{q_i} - \mu_{q'_i} \right) [dx_j] \in H^{1,0} + H_{\mathbb{Z}}^1 \\ \Leftrightarrow \left(\Pi \left(\int dz_{\nu}; \int (\mu_{q_i} - \mu_{q'_i}) \right) \right)_{\nu} \equiv 0 \pmod{\Omega \mathbb{Z}^{2g}}. \end{aligned}$$

By the reciprocity law for differentials of the third kind (cf. [GH78]), we find as $(\mu_{q_i} - \mu_{q'_i})$ is meromorphic with simple poles: $\Leftrightarrow \sum_i (q_i - q'_i) = 0 \in \text{Pic}^0 \bar{X}$. That proves the lemma. \square

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